

# A note on zero-sum 5-flows in regular graphs

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## Abstract

Let  $G$  be a graph. A *zero-sum flow* of  $G$  is an assignment of non-zero real numbers to the edges such that the sum of the values of all edges incident with each vertex is zero. Let  $k$  be a natural number. A *zero-sum  $k$ -flow* is a flow with values from the set  $\{\pm 1, \dots, \pm(k-1)\}$ . It has been conjectured that every  $r$ -regular graph,  $r \geq 3$ , admits a zero-sum 5-flow. In this paper we give an affirmative answer to this conjecture, except for  $r = 5$ .

## 1. Introduction

Nowhere-zero flows on graphs were introduced by Tutte [7] in 1949 and since then have been extensively studied by many authors. A great deal of research in the area has been motivated by Tutte's 5-Flow Conjecture which states that every 2-edge connected graph can have its edges

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directed and labeled by integers from  $\{1, 2, 3, 4\}$  in such a way that Kirchhoffs current law is satisfied at each vertex. In 1983, Bouchet [4] generalized this concept to bidirected graphs. A *bidirected graph*  $G$  is a graph with vertex set  $V(G)$  and edge set  $E(G)$  such that each edge is oriented as one of the four possibilities:  $\bullet \longleftrightarrow \bullet$ ,  $\bullet \rightarrow \bullet$ ,  $\bullet \leftarrow \bullet$ ,  $\bullet \leftrightarrow \bullet$ . Let  $G$  be a bidirected graph. For every  $v \in V(G)$ , the set of all edges with tails (respectively, heads) at  $v$  is denoted by  $E^+(v)$  (respectively,  $E^-(v)$ ). The function  $f : E(G) \rightarrow \mathbb{R}$  is a *bidirected flow* of  $G$  if for every  $v \in V(G)$ , we have

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).$$

If  $f$  takes its values from the set  $\{\pm 1, \dots, \pm(k-1)\}$ , then it is called a *nowhere-zero bidirected  $k$ -flow*.

Consequently, Bouchet proposed the following interesting conjecture.

**Bouchet's Conjecture.** [4, 8] *Every bidirected graph that has a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow.*

Bouchet proved that his conjecture is true if 6 is replaced by 216. Then Zyka reduced 216 to 30 [9].

Let  $G$  be a graph. A *zero-sum flow* for  $G$  is an assignment of non-zero real numbers to the edges such that the sum of the values of all edges incident with each vertex is zero. Let  $k$  be a natural number. A *zero-sum  $k$ -flow* is a flow with values from the set  $\{\pm 1, \dots, \pm(k-1)\}$ . The following conjecture was posed on the zero-sum flows in graphs.

**Zero-Sum Conjecture (ZSC).** [1] *If  $G$  is a graph with a zero-sum flow, then  $G$  admits a zero-sum 6-flow.*

The following conjecture is an improved version of ZSC for regular graphs.

**Conjecture A.** [2] *Every  $r$ -regular graph ( $r \geq 3$ ) admits a zero-sum 5-flow.*

Recently, in connection with this conjecture the following two theorems were proved.

**Theorem 1.** [1] *Let  $r$  be an even integer with  $r \geq 4$ . Then every  $r$ -regular graph has a zero-sum 3-flow.*

**Theorem 2.** [2] *Let  $G$  be an  $r$ -regular graph. If  $r$  is divisible by 3, then  $G$  has a zero-sum 5-flow.*

**Remark 1.** There are some regular graphs with no zero-sum 4-flow. To see this consider the graph given in Figure 1. To the contrary assume this the graph has a zero-sum 4-flow. Since the sum of the values of all edges incident with each vertex is zero, for every  $v \in V(G)$ ,  $-2$  or  $2$  should appear in the neighborhood of  $v$ . On the other hand two numbers with absolute value 2 can not appear in the neighborhood of a vertex. So all edges of  $G$  with values  $\pm 2$  form a perfect matching. But by celebrated Tutte's Theorem [3, p.76],  $G$  has no perfect matching, a contradiction.

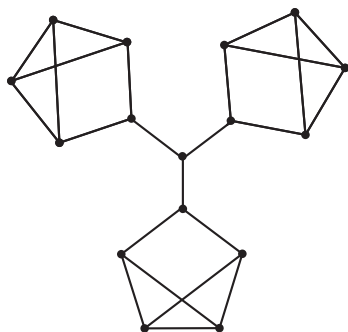


Figure 1. A 3-regular graph with no zero-sum 4-flow

In 2010, the following result was proved.

**Theorem 3.** [2] *Bouchet's Conjecture and ZSC are equivalent.*

Motivated by Bouchet's Conjecture and along with Theorem 3 we focused our attention to establish the Conjecture A. We show that except  $r = 5$ , Conjecture A is true.

## 2. The Main Result

In this section we prove that every  $r$ -regular graph,  $r \geq 3$ ,  $r \neq 5$ , admits a zero-sum 5-flow. Before establishing our main result we need some notations and definitions.

Let  $G$  be a finite and undirected graphs with vertex set  $V(G)$  and edge set  $E(G)$ , where multiple edges and loops are admissible. A  $k$ -regular graph is a graph where each vertex is of degree  $k$ . A subgraph  $F$  of a graph  $G$ , is a *factor* of  $G$  if  $F$  is a spanning subgraph of  $G$ . If a factor  $F$  has all of its degrees equal to  $k$ , it is called a  $k$ -factor. Thus a 2-factor is a disjoint union of finitely

many cycles that cover all the vertices of  $G$ . A  $k$ -factorization of  $G$  is a partition of the edges of  $G$  into disjoint  $k$ -factors. For integers  $a$  and  $b$ ,  $1 \leq a \leq b$ , an  $[a, b]$ -factor of  $G$  is defined to be a factor  $F$  of  $G$  such that  $a \leq d_F(v) \leq b$ , for every  $v \in V(G)$ . For any vertex  $v \in V(G)$ , let  $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ .

The following two theorems are also needed.

**Theorem 4.** [6] *Every  $2k$ -regular multigraph admits a 2-factorization.*

**Theorem 5.** [5] *Let  $r \geq 3$  be an odd integer and let  $k$  be an integer such that  $1 \leq k \leq \frac{2r}{3}$ . Then every  $r$ -regular graph has a  $[k-1, k]$ -factor each component of which is regular.*

**Lemma 1.** *Let  $G$  be an  $r$ -regular graph. Then for every even integer  $q$ ,  $2r \leq q \leq 4r$ , there exists a function  $f : E(G) \rightarrow \{2, 3, 4\}$  such that for every  $u \in V(G)$ ,  $\sum_{v \in N_G(u)} f(uv) = q$ .*

**Proof.** First assume that  $r$  is an odd integer. For every edge  $e = uv$ , we add a new edge  $e' = uv$  to the graph  $G$  and call the resultant graph by  $G'$ . Clearly,  $G'$  is a  $2r$ -regular multigraph. By Theorem 4,  $G'$  admits a 2-factorization with 2-factors  $F_1, \dots, F_r$ . Now, for every  $e \in F_i$ ,  $1 \leq i \leq r$ , we define a function  $g : E(G') \rightarrow \{1, 2\}$  as follows:

$$g(e) = \begin{cases} 2, & 1 \leq i \leq \frac{q-2r}{2}; \\ 1, & \frac{q-2r}{2} < i. \end{cases}$$

Therefore for each  $v \in V(G')$ ,  $\sum_{v \in N_{G'}(u)} g(uv) = q$ . Now, define a function  $f : E(G) \rightarrow \{2, 3, 4\}$  such that for every  $e = uv \in E(G)$ ,  $f(e) = g(e) + g(e')$ , where  $e' = uv$  in  $G'$ . Then for every  $u \in V(G)$ ,  $\sum_{v \in N_G(u)} f(uv) = q$ , as desired.

Now, let  $r$  be an even integer. Since  $G$  is an  $r$ -regular graph, by Theorem 4,  $G$  admits a 2-factorization with 2-factors  $F_1, \dots, F_{\frac{r}{2}}$ . Now, for every  $e \in F_i$ ,  $1 \leq i \leq \frac{r}{2}$ , we define a function  $f : E(G) \rightarrow \{2, 3, 4\}$  as follows:

$$f(e) = \begin{cases} 4, & 1 \leq i \leq \lfloor \frac{q-2r}{4} \rfloor; \\ 3, & \lfloor \frac{q-2r}{4} \rfloor < i \leq \lceil \frac{q-2r}{4} \rceil; \\ 2, & \lceil \frac{q-2r}{4} \rceil < i. \end{cases}$$

It is not hard to verify that for every  $u \in V(G)$ ,  $\sum_{v \in N_G(u)} f(uv) = q$ , as desired.  $\square$

Now, we are in a position to prove our main theorem.

**Theorem 6.** *Let  $r \geq 3$  and  $r \neq 5$ . Then every  $r$ -regular graph has a zero-sum 5-flow.*

**Proof.** First we prove the theorem for  $r = 7$ . Let  $G$  be a 7-regular graph. Then by Theorem 5,  $G$  has a  $[3, 4]$ -factor, say  $H$ , whose components are regular. Let  $H_1$  be the union of the 3-regular components of  $H$  and let  $H_2$  be the union of 4-regular components of  $H$ . By Theorem 4,  $H_2$  can be decomposed into two 2-factors  $H'_2$  and  $H''_2$ . Assign 1 and 2 to all edges of  $H'_2$  and  $H''_2$ , respectively. By Lemma 1, there exists a function  $f : E(H_1) \rightarrow \{2, 3, 4\}$  such that for every  $u \in V(H_1)$ ,  $\sum_{v \in N_{H_1}(u)} f(uv) = 8$ . Now, assign  $-2$  to every edge in  $E(G) \setminus E(H)$  and we are done.

Now, let  $r \geq 9$  be an odd integer. By Theorem 5, for every  $k$ ,  $k \leq \frac{2r}{3}$ ,  $G$  has a  $[k-1, k]$ -factor whose components are regular. Let  $k = \lfloor \frac{2r}{3} \rfloor$ ,  $k' = r - k$ , and  $H$  be a  $[k-1, k]$ -factor of  $G$  such that  $H_1$  be the union of  $(k-1)$ -regular subgraph of  $H$  and  $H_2 = H \setminus H_1$ . It can be easily checked that  $k \leq 2k' \leq 2k - 4$ . Hence by Lemma 1, there exists a function  $f : E(H_1) \rightarrow \{2, 3, 4\}$  such that for every  $u \in V(H_1)$ ,  $\sum_{v \in N_{H_1}(u)} f(uv) = 4k' + 4$ . Also by Lemma 1, there exists a function  $f : E(H_2) \rightarrow \{2, 3, 4\}$  such that for every  $v \in V(H_2)$ ,  $\sum_{u \in N_{H_2}(v)} f(uv) = 4k'$ . Finally assign  $-4$  to every edge of  $E(G) \setminus E(H)$ . Now, by Theorem 1 and Theorem 2 the proof is complete.  $\square$

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